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# Langevin dynamics, stochastic quantization and the supersymmetric $1/r^2$ systems

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**Abstract.** We establish a novel correspondence between the spacetime correlators of one-dimensional (1D) $N$-particle classical stochastic models described by a Langevin equation with that of the ground-state dynamics of a class of integrable 1D interacting many-body quantum models of the *supersymmetric* elliptic $1/r^2$ type. We show that these seemingly different concepts of stochastic systems, supersymmetry and quantum integrability can be viewed in a unified framework. Starting with an $N \times N$ Lax matrix, we show that row (column) sums driven by a Gaussian noise term may be interpreted as a set of forward (backward) Langevin equations. Then, following functional path integral methods of stochastic quantization, we straightforwardly find an associated supersymmetric 1D quantum Hamiltonian. If, further, the classical stochastic system consists of two-body interactions only and we also want the quantum interactions to be of two-body type, we find that the only class of interactions permissible for the quantum models corresponds to the elliptic $1/r^2$ models. The algebraic structure that emerges very naturally reproduces the proof of integrability and allows the identification of the ground-state wavefunction of these quantum models.

## 1. Introduction

In 1971, Sutherland [1] made an intriguing discovery. The square of the ground-state wavefunction of a one-dimensional many-particle system interacting via the pairwise potential $\lambda(\lambda-1)/\sin^2(x)$ is identical to the joint probability density function for the eigenvalues of matrices from Dyson's circular ensemble [2], $\lambda = \frac{1}{2}$, 1 and 2 corresponding to orthogonal, unitary and symplectic ensembles, respectively. This connection with the theory of random matrices then enables the explicit calculation of various non-trivial static correlation functions such as, for example, the one-particle reduced density matrix [1, 3] for all length scales.

Until recently, the underlying reasons for this above connection of an exactly soluble quantum many-body system and disordered or quantum chaotic systems had not been exposed. However, in a series of remarkable works, Altshuler, Simons and co-workers [4] have calculated a certain correlation function of two variables by the supersymmetric method of Efetov [5] which after suitable rescaling of the parameters is just a dynamical correlation function of the $1/\sin^2(x)$ model. Subsequent work done by various authors has enabled the calculation of other dynamical correlation functions [6] and clarified the route from a two random matrix problem to $1/r^2$ type models [7].

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In this work, we establish a connection between classical stochastic systems (described by a Langevin or a Fokker–Planck (FP) equation) and the *supersymmetric* $1/r^2$ type models by stochastically quantizing the classical systems [8]. We show that if the interaction potentials of the stochastic system and the resulting quantum system are chosen to be pairwise and translationally invariant, then the only class of interactions possible for the quantum models corresponds to the elliptic $1/r^2$ type considered by Calogero and Sutherland [1, 9, 10]. The algebraic structure that emerges very naturally allows for the proof of integrability of these quantum models as shown in [11]. Finally, we show how the Langevin description, the supersymmetry and the integrability can be reconstructed by using suitable sums and products of Fermi creation and annihilation operators and the Lax $l$ matrix.

## 2. Parisi–Wu quantization of an $N$-particle gas

Let $S[x]$ denote the Euclidean 'action' of an $N$-particle system with coordinate set $\{x\} = \{x_1, \ldots, x_N\}$. We now introduce a time parameter $\tau$ and consider the following set of Langevin equations for the behaviour of the $N$ interacting particles,

$$\frac{\mathrm{d}x_j}{\mathrm{d}\tau} = -\frac{\partial S}{\partial x_j} + \eta_j(\tau). \tag{1}$$

Here $\eta$ is a Gaussian random variable representing the noise driving the Langevin equation, i.e. $\langle \eta_j \rangle_\eta = 0$, $\langle \eta_j(\tau)\eta_k(0)\rangle_\eta = 2\delta_{jk}\delta(\tau)$. The angular brackets denote a connected average with respect to the random variable $\eta$. This implies that the probability distribution of the noise is given as

$$\mathcal{P}[\eta] = \exp\left[-\frac{1}{4}\sum_{j=1}^{N}\int_0^\infty \mathrm{d}\tau\, \eta_j^2(\tau)\right] \tag{2}$$

normalized as $\int \mathcal{D}[\eta]\mathcal{P}[\eta] = 1$ with discrete time slices $\tau_m = m\epsilon$ and $\mathcal{D}[\eta] = \prod_j^N \prod_{m=1}^\infty D[\eta_j(\tau_m)]$.

Following Gozzi [12], we now consider the generating functional for performing the noise and the initial configuration average. It is defined as

$$Z[J] = \int \mathcal{D}[\eta] \int \mathrm{d}[x(0)] P[x(0)] \exp\left[-\frac{1}{4}\sum_{j=1}^{N}\int_0^\infty \mathrm{d}\tau\, \eta_j^2(\tau) - \sum_{j=1}^{N}\int_0^\infty \mathrm{d}\tau\, J_j x_j^\eta\right] \tag{3}$$

with $\mathrm{d}[x(0)] = \prod_{j=1}^N \mathrm{d}x_j(0)$. Further, $x_j^\eta$ denotes the solution of the Langevin equations for a given realization of the noise and with some initial probability distribution of the particles $P[x(0)]$. $\{J\} = (J_1, J_2, \ldots, J_N)$ are the source terms. Note that we have already normalized the generating functional such that $Z[0] = 1$. The time-dependent correlation functions can be derived by differentiating the generating function appropriately:

$$\langle x_1^\eta(\tau_1)x_2^\eta(\tau_2)\ldots x_n^\eta(\tau_n)\rangle_\eta = \frac{(-1)^n\delta^n Z[J]}{\delta J_1(\tau_1)\delta J_2(\tau_2)\ldots\delta J_n(\tau_n)}. \tag{4}$$

We now use the Langevin equation (1) to make the following Nicolai mapping [13] for $\eta$ in $Z$:

$$\eta_j(\tau) \to \frac{\mathrm{d}x_j}{\mathrm{d}\tau} + \frac{\partial S}{\partial x_j}. \tag{5}$$

The Jacobian of this transformation is given as $\det[\delta\eta_j(\tau)/\delta x_k(\tau')]$ and we may evaluate straightforwardly

$$\frac{\delta\eta_j(\tau)}{\delta x_k(\tau')} = \left[\delta_{jk}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{\partial^2 S[x]}{\partial x_j^2}\right\} + (1-\delta_{jk})\frac{\partial^2 S[x]}{\partial x_k \partial x_j}\right]\delta(\tau-\tau'). \tag{6}$$

We next introduce Grassman variables $c_j(\tau)$, $c_j^*(\tau)$ and use the identity

$$\det[M] = \int \mathcal{D}[c^*]\mathcal{D}[c]\exp\left[-\int_0^\infty \mathrm{d}\tau\,\mathrm{d}\tau' \sum_{jk} c_j^*(\tau)M_{jk}(\tau,\tau')c_k(\tau')\right] \tag{7}$$

for the Jacobian. We then have for $Z[J]$,

$$\begin{aligned} Z[J] = \int &\mathcal{D}[x]\mathcal{D}[c^*]\mathcal{D}[c]P[x(0)] \\ &\times \exp - \int \mathrm{d}\tau \left\{ \sum_j \frac{1}{4}\left[\frac{\mathrm{d}x_j}{\mathrm{d}\tau} + \frac{\partial S[x]}{\partial x_j}\right]^2 + \sum_j c_j^*\left[\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{\partial^2 S[x]}{\partial x_j^2}\right]c_j \right. \\ &\left. + \sum_{j\neq k} \frac{\partial^2 S[x]}{\partial x_k \partial x_j} c_j^* c_k + \sum_j J_j x_j \right\}. \end{aligned} \tag{8}$$

We now rescale the Euclidean time $\tau \to 2\tau$ and open the square in the first term of the exponential in equation (8). The resulting cross term can be integrated explicitly, i.e.

$$\int_0^\infty \mathrm{d}\tau \sum_j \frac{\mathrm{d}x_j}{\mathrm{d}\tau}\frac{\partial S[x]}{\partial x_j} = \int_0^\infty \mathrm{d}\tau \frac{\mathrm{d}S[x]}{\mathrm{d}\tau} = S[x(\infty)] - S[x(0)]. \tag{9}$$

We then get

$$\begin{aligned} Z[J] = \int &\mathcal{D}[c^*]\mathcal{D}[c]\mathrm{d}[x(0)]P[x(0)]\,\mathrm{e}^{S[x(0)]/2}D[x(\infty)]\,\mathrm{e}^{-S[x(\infty)]/2}\mathcal{D}[x'] \\ &\times \exp - \int \mathrm{d}\tau \left(\mathcal{L} + \sum_j J_j x_j\right) \end{aligned} \tag{10}$$

where

$$\mathcal{D}[x'] = \prod_{j=1}^N \lim_{n\to\infty} \prod_{m=1}^{n-1} D[x_j(\tau_m)] \tag{11}$$

and the Lagrangian is given as

$$\mathcal{L} = \sum_j \left[\frac{1}{2}\left(\frac{\mathrm{d}x_j}{\mathrm{d}\tau}\right)^2 + \frac{1}{8}\left(\frac{\partial S[x]}{\partial x_j}\right)^2\right] + \sum_j c_j^*\left[\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{1}{2}\frac{\partial^2 S[x]}{\partial x_j^2}\right]c_j + \sum_{j\neq k}\frac{1}{2}\frac{\partial^2 S[x]}{\partial x_k \partial x_j}c_j^* c_k. \tag{12}$$

Let us now choose an initial distribution $P[x(0)] = \delta(x(\infty) - x(0))$. This choice cancels the boundary term in the integral in the exponent. The generating functional is then given as

$$Z[\{J\}] = \int \mathcal{D}[c^*]\mathcal{D}[c]\mathcal{D}[x'']\exp - \int \mathrm{d}\tau \left(\mathcal{L} + \sum_j J_j x_j\right) \tag{13}$$

with $\mathcal{D}[x''] = \mathcal{D}[x']D[x(\infty)]$. Casting $\mathcal{L}$ in a form symmetric in $c^*$ and $c$, we get

$$\mathcal{L} = \sum_j \left[\frac{1}{2}\left(\frac{\mathrm{d}x_j}{\mathrm{d}\tau}\right)^2 + \frac{1}{8}\left(\frac{\partial S[x]}{\partial x_j}\right)^2\right] + \sum_j \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau}[c_j^*, c_j] + \sum_j \frac{1}{4}\frac{\partial^2 S[x]}{\partial x_j^2}[c_j^*, c_j]$$

$$+\sum_{j\neq k}\frac{1}{4}\frac{\partial^2 S[x]}{\partial x_k \partial x_j}[c_j^*, c_k]. \tag{14}$$

Next, we Wick-rotate the Euclidean Lagrangian by $\tau \rightarrow \mathrm{i}t$. We introduce position and momentum operators with canonical commutation relations

$$[\hat{x}_j, \hat{p}_k] = \mathrm{i}\delta_{jk}. \tag{15}$$

The fermionic degrees of freedom from the treatment of the Jacobian are taken into account by the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ and

$$\{\hat{\psi}_j^\dagger, \hat{\psi}_k\} = \delta_{jk}. \tag{16}$$

Then the Hamiltonian corresponding to the Lagrangian $\mathcal{L}$ is

$$H = \sum_j \left[\frac{\hat{p}_j^2}{2} + \frac{1}{8}\left(\frac{\partial S[x]}{\partial x_j}\right)^2 + \frac{1}{4}\frac{\partial^2 S[x]}{\partial x_j^2}[\hat{\psi}_j^\dagger, \hat{\psi}_j]\right] + \sum_{j\neq k}\frac{1}{4}\frac{\partial^2 S[x]}{\partial x_j \partial x_k}[\hat{\psi}_j^\dagger, \hat{\psi}_k]. \tag{17}$$

Opening the commutator brackets, we find that $H$ can be written as a sum of a purely bosonic and a purely fermionic part as $H = H_\mathrm{b} + H_\mathrm{f}$, with

$$H_\mathrm{b} = \sum_j \frac{\hat{p}_j^2}{2} + \frac{1}{8}\left(\frac{\partial S[x]}{\partial x_j}\right)^2 - \frac{1}{4}\frac{\partial^2 S[x]}{\partial x_j^2} \tag{18}$$

and

$$H_\mathrm{f} = \sum_j \frac{1}{2}\frac{\partial^2 S[x]}{\partial x_j^2}\hat{\psi}_j^\dagger\hat{\psi}_j + \sum_{j\neq k}\frac{1}{2}\frac{\partial^2 S[x]}{\partial x_j \partial x_k}\hat{\psi}_j^\dagger\hat{\psi}_k. \tag{19}$$

Note here that the use of the terms bosonic and fermionic is slightly misleading, since the system described by $H_\mathrm{b}$ can also include fermionic degrees of freedom.

The choice of initial condition of the probability distribution as $P[x(0)] = \delta(x(\infty) - x(0))$ puts periodic boundary conditions on the bosonic path integral. In evaluating the determinant in equation (7), periodic boundary conditions are imposed on the fermionic degrees. These two boundary conditions together project out only states with initial particle distribution equal to the equilibrium distribution $P[x(0)] = \mathrm{e}^{-S[x_0]}$ and zero fermion occupation number. This then implies that the ground state of the Hamiltonian $H$ is simply given as

$$\Psi_\mathrm{g} = \mathcal{N}\,\mathrm{e}^{-S/2}|0\rangle \tag{20}$$

where $|0\rangle$ denotes the Fermi vacuum and $\mathcal{N}$ is a normalization constant. Hence by studying the equilibrium distribution of the classical stochastic system, the ground state of a highly non-trivial interacting many-body quantum system can be obtained. Furthermore, the correlation functions $\langle x_1^\eta(\tau_1)x_2^\eta(\tau_2)\ldots x_n^\eta(\tau_n)\rangle_\eta$ of equation (4) are stationary, i.e. they are functions of the differences $\tau_j - \tau_k$ only.

## 3. Hidden symmetries and the two-body condition

As has been noted previously in [12], the Hamiltonian $H$ exhibits a hidden non-relativistic variant of supersymmetry. We define the quantities

$$Q_j = \hat{p}_j - \frac{\mathrm{i}}{2}\frac{\partial S[x]}{\partial x_j} \tag{21}$$

$$\zeta = \sum_j Q_j\hat{\psi}_j^\dagger. \tag{22}$$

Then it immediately follows by construction that $\zeta\Psi_g = \zeta^\dagger\Psi_g = 0$ and that $[Q_j, Q_k] = 0$, and

$$H = \tfrac{1}{2}\{\zeta, \zeta^\dagger\} \tag{23}$$

$$= \tfrac{1}{2}\sum_j Q_j^\dagger Q_j + \tfrac{1}{2}\sum_{jk}[Q_j, Q_k^\dagger]\hat{\psi}_j^\dagger\hat{\psi}_k. \tag{24}$$

Furthermore, since $\zeta^2 = (\zeta^\dagger)^2 = 0$, we also have that $[H, \zeta] = [H, \zeta^\dagger] = 0$. We note that this quantum mechanical supersymmetry has been studied in detail in [14]

Let us now make the ansatz that the action $S$ that appeared in the Langevin equation (1) should consist of a sum of translationally invariant two-body interactions, i.e.

$$S[x] = 2\sum_{j\neq k} s(x_j - x_k) \tag{25}$$

with $s(x) = s(-x)$. We further define $\sigma_{jk} = \sigma(x_j - x_k) = \partial s(x_j - x_k)/\partial x_j$. We now want to make contact between the above procedure and explicit models of one-dimensional quantum systems. Thus we look for a translationally invariant two-body potential $V(|x_j - x_k|)$, such that

$$\sum_{j\neq k} V(|x_j - x_k|) - E_0 = \sum_j \frac{1}{8}\left(\frac{\partial S[x]}{\partial x_j}\right)^2 - \frac{1}{4}\frac{\partial^2 S[x]}{\partial x_j^2} \tag{26}$$

$$= 2\sum_{j\neq k}\sigma_{jk}^2 - \sum_{j\neq k}\sigma'_{jk} + 2\sum_{jkl}\sigma_{jk}\sigma_{jl}. \tag{27}$$

The first two terms on the right-hand side of equation (27) are already in the required form. The third term on the right-hand side may be rewritten as a sum over triples, i.e. $\sigma_{jk}\sigma_{kl} + \sigma_{kl}\sigma_{lj} + \sigma_{lj}\sigma_{jk}$. With $a = x_j - x_k$, $b = x_k - x_l$ and $c = x_l - x_j = -a - b$, we are then looking for some odd function $\sigma(x)$, such that

$$\sigma(a)\sigma(b) + \sigma(b)\sigma(c) + \sigma(c)\sigma(a) = v(a) + v(b) + v(c) \tag{28}$$

with $a + b + c = 0$ and $v(x)$ even. This equation has already appeared in the literature in the context of the $1/r^2$ type potentials [1, 9, 10] and we may immediately conclude the following.

(i) The most general solution of the above equation for a periodic potential is given by

$$\sigma(x) = -\lambda Z(x|n) \tag{29}$$

$$s(x) = -\ln[\theta^\lambda(x|n)] \tag{30}$$

where $Z(x|n)$ and $\theta(x|n)$ are the Jacobi zeta and theta function with modulus $n$, respectively. Taking suitable limits, this then includes the hyperbolic potential $V_h = \lambda(\lambda-1)/\sinh^2(x)$ corresponding to the action $S_h[x] = -\lambda\sum_{j\neq k}\ln|\sinh[x_j - x_k]|$, the trigonometric potential $V_t = \lambda(\lambda-1)/\sin^2(x)$ corresponding to the action $S_t[x] = -\lambda\sum_{j\neq k}\ln|\sin[x_j - x_k]|$ and, lastly, the potential $V_0 = \lambda(\lambda-1)/x^2$ with $S_0[x] = -\lambda\sum_{j\neq k}\ln|x_j - x_k|$.

(ii) The ground state of the supersymmetric Hamiltonian is

$$\Psi = \mathcal{N}\prod_{j<k}^{N}[\theta(x_j - x_k|n)]^\lambda|0\rangle \tag{31}$$

in agreement with equation (20).

We emphasize that the supersymmetric $1/r^2$ type Hamiltonians therefore constitute the *only* possible family of translationally invariant two-body quantum Hamiltonians associated with stochastic overdamped two-body systems described by a Langevin equation. Note also that this choice of action $S$ to consist of pairwise interactions only, restricts the possible ground state to be of the Jastrow type.

## 4. The Lax $l$ and $m$ matrices

Let us now back up a little and see if we can find a unifying theme for the above constructions. We then remember that the bosonic Hamiltonian is well known to be integrable [10] by a technique due to Lax [15] which replaces the equations of motion by a matrix commutator of two matrices $l$ and $m$. Integrals of motion can be constructed as $J_n = \mathrm{Tr}[l^n \Lambda]$ with $\Lambda_{jk} = 1$ for all $j, k$ [11]. The Hamiltonian is included in this series of integrals since $J_2 = H - E_0$. This indicates that one should in fact view the $l$ matrix and not the Hamiltonian as the fundamental quantity in these models. Therefore, let us introduce the Lax $l$ matrix given as

$$l_{jk} = \hat{p}_j \delta_{jk} + \mathrm{i}\sigma_{jk}(1 - \delta_{jk}). \tag{32}$$

Keeping in mind the earlier Wick-rotation and the rescaling $\tau \to 2\tau$, and further replacing the operators by their classical observables, we find that the Langevin equation (1) may be written as

$$\sum_k l_{jk} = Q_j = \eta_j(\tau). \tag{33}$$

We remark that, instead of the row sums of $l$, we could have also used the column sums. This would only have changed the direction of time. Equation (33) also establishes a direct connection of the Lax $l$ matrix with the generators of supersymmetry.

The integrability of the bosonic Hamiltonian $H_\mathrm{b}$ was shown by Shastry and Sutherland [11] by defining the operator $L = \sum_{jk} l_{jk} \hat{\psi}_j^\dagger \hat{\psi}_k$, which satisfies the commutation relation $[H, L] = 0$. Note that this equation in fact implies the above-mentioned Lax relation for the bosonic system, i.e. $[l, H_\mathrm{b}] = ml - lm$. The $m$ matrix is simply contained in the fermionic Hamiltonian as $H_\mathrm{f} = \sum_{jk} m_{jk} \hat{\psi}_j^\dagger \hat{\psi}_k$. The proof of the integrability of the system described by the bosonic Hamiltonian further requires the row and column sums of the $m$ matrix to vanish [11]. An important feature of the algebraic structure provided by our construction is the origin of the $m$ matrix in the fermionic part of the Hamiltonian $H$. Further, it automatically satisfies the above row and column sum constraint.

Adding a one-body term to the form of $S$, we may again follow the above route from equation (1) to equation (24). Thus the supersymmetric generalizations of the Calogero type potentials [16] with one-body terms $\omega x_j^2/2$ are included in the present derivation. However, for finite $\omega$, the proof of integrability no longer holds for these systems.

## 5. Conclusions

Let us summarize. We start with an $N \times N$ matrix $l$ such that its diagonal elements correspond to the momenta of a classical $N$-particle system and the off-diagonal elements are odd functions of the coordinates specified by row and column indices only. Next, we sum over the rows of this matrix and drive the resulting differential equations with a Gaussian random force. This Langevin equation is then treated by means of the Parisi–Wu stochastic quantization, resulting in the construction of a Hamiltonian for an associated quantum $N$-particle system at $T = 0$. This quantum system is necessarily supersymmetric, the generators of the symmetry being again the row (column) sums of the $l$ matrix together with simple Fermi operators. If we now restrict the interactions of our quantum system to be of the two-body type, we find that elliptic interactions give the most general possible solution.

This connection of a Langevin equation with models of the $1/r^2$ type has been established previously by Dyson in the context of his studies of the so-called log–sine gas [2].

There, the $1/\sin^2(x)$ Hamiltonian is simply the Euclideanized Fokker–Plank Hamiltonian. Here, we show in addition that any translationally invariant two-body interaction in the Langevin description necessarily restricts the associated two-body quantum system to belong to the family of $1/r^2$ models. In this sense, the $1/r^2$ model is the universal model of disordered overdamped systems described by a Langevin equation.

Furthermore, the Lax $m$ matrix, which is necessary for the Lax-type proof of integrability of the bosonic quantum system described by $H_\mathrm{b}$, appears very naturally in the fermionic part $H_\mathrm{f}$ of the supersymmetric Hamiltonian $H$.

In closing, we note that, unfortunately, our method does not seem to be useful for the calculation of correlation functions of either the classical disordered system or the associated quantum system. It does, however, reveal a fascinating connection of integrability and supersymmetry via the route of stochastic quantization.

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## References

[1] Sutherland B 1971 *Phys. Rev.* A **4** 2019
[2] Dyson F J 1962 *J. Math. Phys.* **3** 140, 157, 166, 1191
[3] Sutherland B 1992 *Phys. Rev.* B **45** 907
Römer R A and Sutherland B 1993 *Phys. Rev.* B **48** 6058
[4] Simons B D and Altshuler B L 1993 *Phys. Rev. Lett.* **70** 4063
Simons B D, Lee P A and Altshuler B L 1993 *Phys. Rev. Lett.* **70** 4122
Simons B D, Szafer A and Altshuler B L 1993 *JETP Lett.* **57** 276
[5] Efetov K B 1983 *Adv. Phys.* **32** 53
[6] Haldane F D M and Zirnbauer M R 1993 *Phys. Rev. Lett.* **71** 4055
Haldane F D M 1994 *Proc. 16th Taniguchi Int. Symp. on the Theory of Condensed Matter: Correlation Effects in Low Dimensional Electron Systems* ed N Kawakami and A Okiji (Berlin: Springer)
Deo N, Jain S and Shastry B S 1994 *Preprint* IISc-CTS; 1995 *Preprint* IISc-CTS
[7] Narayan O and Shastry B S 1993 *Phys. Rev. Lett.* **71** 2106
Shastry B S 1994 *Proc. 16th Taniguchi Int. Symp. on the Theory of Condensed Matter: Correlation Effects in Low Dimensional Electron Systems* ed N Kawakami and A Okiji (Berlin: Springer)
[8] Parisi G and Wu-Yong-Shi 1981 *Sci. Sin.* **24** 484
[9] Calogero F 1975 *Lett. Nuovo Cimento* **13** 411, 507
[10] Sutherland B 1985 *Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory* (Lecture Notes in Physics **242**) ed B S Shastry, S S Jha and V Singh (Berlin: Springer)
[11] Shastry B S and Sutherland B 1993 *Phys. Rev. Lett.* **70** 4029
[12] Gozzi E 1983 *Phys. Rev.* D **28** 1922
[13] Nicolai H 1980 *Phys. Lett.* **89B** 341; *Nucl. Phys.* B **176** 419
[14] Freedman D Z and Mende P F 1990 *Nucl. Phys.* B **344** 317
[15] Lax P D 1968 *Commun. Pure Appl. Math.* **21** 467
[16] Calogero F 1969 *J. Math. Phys.* **10** 2191
Sutherland B 1971 *J. Math. Phys.* **12** 246